

Triangulated categories of Gorenstein cyclic quotient singularities

Kazushi Ueda

Abstract

We prove an equivalence of triangulated categories between Orlov's triangulated category of singularities for a Gorenstein cyclic quotient singularity and the derived category of representations of a quiver with relations which is obtained from the McKay quiver by removing one vertex and half of the arrows.

Fix an integer n greater than one. For a finite subgroup G of $GL_n(\mathbb{C})$, let $\{\rho_i\}_{i=0}^{N-1}$ be the set of irreducible representations of G . Let further ρ_{Nat} be the natural n -dimensional representation of G given by the inclusion. For $k, l = 0, \dots, N-1$, define the natural numbers a_{kl} by the decomposition

$$\rho_l \otimes \rho_{\text{Nat}} = \bigoplus_k \rho_k^{\oplus a_{kl}}$$

of the tensor product into the direct sum of irreducible representations. The *McKay quiver* of G is the quiver whose set of vertices is $\{\rho_i\}_{i=0}^{N-1}$ and the number of whose arrows from the vertex ρ_k to the vertex ρ_l is a_{kl} [5].

Now assume that G is a cyclic group generated by an element of the form $g = \text{diag}(\zeta^{a_1}, \dots, \zeta^{a_n})$, where a_1, \dots, a_n are positive integers such that $\gcd(a_1, \dots, a_n) = 1$ and $\zeta = \exp[2\pi\sqrt{-1}/(a_1 + \dots + a_n)]$. Let $R = \mathbb{C}[x_1, \dots, x_n]$ be the polynomial ring in n variables equipped with the \mathbb{Z} -grading given by $\deg x_i = a_i$, $i = 1, \dots, n$. Define another \mathbb{Z} -graded ring $A(a_1, \dots, a_n) = \bigoplus_{k \geq 0} A_k$ by

$$A_k = R_{k(a_1 + \dots + a_n)}. \quad (1)$$

Then $A(a_1, \dots, a_n)$ is the invariant ring of R by the action of G so that $\mathbb{C}^n/G = \text{Spec } A(a_1, \dots, a_n)$.

In this case, the corresponding McKay quiver has $N = a_1 + \dots + a_n$ vertices $\{\rho_k\}_{k=0}^{N-1}$ and nN arrows $\{x_{i,k}\}_{\substack{i=1, \dots, n \\ k=0, \dots, N-1}}$, where ρ_k is the one-dimensional representation of G sending $g \in G$ to $\exp[-2k\pi\sqrt{-1}/(a_1 + \dots + a_n)] \in \mathbb{C}^\times$, and $x_{i,k}$ is an arrow from ρ_k to ρ_{k+a_i} .

Next we introduce another quiver $Q(a_1, \dots, a_n)$ obtained by removing the vertex ρ_0 and half of the arrows from the McKay quiver; the set of vertices of $Q(a_1, \dots, a_n)$ is $\{\rho_k\}_{k=1}^N$, and an arrow of the McKay quiver from ρ_k to ρ_l is an arrow of Q_g if $0 < k < l$.

A *quiver with relations* is a pair $\Gamma = (Q, \mathcal{I})$ of a quiver Q and a two-sided ideal \mathcal{I} of its path algebra $\mathbb{C}Q$. We equip $Q(a_1, \dots, a_n)$ with the relations $\mathcal{I}(a_1, \dots, a_n)$ generated by $x_{j,k+a_i}x_{i,k} - x_{i,k+a_j}x_{j,k}$ for $1 \leq i < j \leq n$ and $k = 1, \dots, N - a_i - a_j - 1$, and put $\Gamma(a_1, \dots, a_n) = (Q(a_1, \dots, a_n), \mathcal{I}(a_1, \dots, a_n))$. The main theorem is:

Theorem 1. *For a sequence a_1, \dots, a_n of positive integers such that $\gcd(a_1, \dots, a_n) = 1$, there exists an equivalence*

$$D_{\text{Sg}}^{\text{gr}}(A(a_1, \dots, a_n)) \cong D^b \text{ mod } \Gamma(a_1, \dots, a_n)$$

of triangulated categories.

Here, $D^b \text{mod } \Gamma(a_1, \dots, a_n)$ is the bounded derived category of finite-dimensional right modules over the path algebra $\mathbb{C}\Gamma(a_1, \dots, a_n) = \mathbb{C}Q(a_1, \dots, a_n)/\mathcal{I}(a_1, \dots, a_n)$ with relations.

$D_{\text{Sg}}^{\text{gr}}(A(a_1, \dots, a_n))$ is the *triangulated category of singularities*, defined by Orlov [6] as the quotient category

$$D_{\text{Sg}}^{\text{gr}}(A(a_1, \dots, a_n)) = D^b \text{gr } A(a_1, \dots, a_n) / D^b \text{grproj } A(a_1, \dots, a_n) \quad (2)$$

of the bounded derived category $D^b \text{gr } A(a_1, \dots, a_n)$ of finitely-generated \mathbb{Z} -graded $A(a_1, \dots, a_n)$ -modules by its full triangulated subcategory $D^b \text{grproj } A(a_1, \dots, a_n)$ consisting of perfect complexes. The $n = 2$ case in the above theorem is due to Takahashi [7] (see also Kajiura, Saito, and Takahashi [4]).

The proof goes as follows: Let

$$\text{qgr } R := \text{gr } R / \text{tor } R$$

be the quotient category of the abelian category $\text{gr } R$ of finitely-generated \mathbb{Z} -graded R -modules by its full subcategory $\text{tor } R$ consisting of torsion modules, and $\pi : \text{gr } R \rightarrow \text{qgr } R$ be the natural projection functor. For $M \in \text{gr } R$ and $l \in \mathbb{Z}$, $M(l)$ denotes the graded R -module shifted by l ; $M(l)_k = M_{l+k}$. Define a shift operator $s : \text{qgr } R \rightarrow \text{qgr } R$ by $s(\pi M) = \pi M(a_1 + \dots + a_n)$ and put $\mathcal{O} = \pi R$. Then one has $A(a_1, \dots, a_n) = \bigoplus_{k=0}^{\infty} \text{Hom}(\mathcal{O}, s^k(\mathcal{O}))$. Since $\gcd(a_1, \dots, a_n) = 1$, the graded R -module $R(l)$ for any $l \in \mathbb{Z}$ is generated up to torsion modules by the subset $\bigcup_{j \in \mathbb{N}} R(l)_{j(a_1 + \dots + a_n)}$ consisting of elements whose degrees are positive multiples of $a_1 + \dots + a_n$. Hence s is ample and one has

$$\text{qgr } R \cong \text{qgr } A(a_1, \dots, a_n)$$

by Artin and Zhang [1, Theorem 4.5].

Since $s^{-1}(\mathcal{O})$ is the dualizing sheaf, $A(a_1, \dots, a_n)$ is Gorenstein with Gorenstein parameter 1 (cf. [6, Lemma 2.11]). Therefore one has a semiorthogonal decomposition

$$D^b \text{qgr } A(a_1, \dots, a_n) = \langle \mathcal{O}, D_{\text{Sg}}^{\text{gr}}(A(a_1, \dots, a_n)) \rangle$$

by Orlov [6, Theorem 2.5(i)]. Here, $D^b \text{qgr } A(a_1, \dots, a_n)$ denotes the bounded derived category of the abelian category $\text{qgr } A(a_1, \dots, a_n)$. On the other hand, $D^b(\text{qgr } R)$ has a full strong exceptional collection $(\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(a_1 + a_2 + \dots + a_n - 1))$ (see e.g. [2, Theorem 2.12]). Hence $D_{\text{Sg}}^{\text{gr}}(A(a_1, \dots, a_n))$ is equivalent to the full triangulated subcategory of $D^b \text{qgr } A(a_1, \dots, a_n)$ generated by the exceptional collection $(\mathcal{O}(1), \dots, \mathcal{O}(a_1 + \dots + a_n - 1))$. By Bondal [3, Theorem 6.2], this subcategory is isomorphic to the derived category of finite-dimensional right modules over the total morphism algebra

$$\bigoplus_{i,j=1}^{a_1 + \dots + a_n - 1} \text{Hom}(\mathcal{O}(i), \mathcal{O}(j))$$

of this collection, which is isomorphic to $\mathbb{C}\Gamma(a_1, \dots, a_n)$.

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References

- [1] M. Artin and J. J. Zhang. Noncommutative projective schemes. *Adv. Math.*, 109(2):228–287, 1994.
- [2] D. Auroux, L. Katzarkov, and D. Orlov. Mirror symmetry for weighted projective planes and their noncommutative deformations. *math.AG/0404281*, 2004.
- [3] A. I. Bondal. Representations of associative algebras and coherent sheaves. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(1):25–44, 1989.
- [4] H. Kajiura, K. Saito, and A. Takahashi. Matrix factorizations and representations of quivers II: type ADE case. *math.AG/0511155*, 2005.
- [5] J. McKay. Graphs, singularities, and finite groups. In *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*, volume 37 of *Proc. Sympos. Pure Math.*, pages 183–186. Amer. Math. Soc., Providence, R.I., 1980.
- [6] D. Orlov. Derived categories of coherent sheaves and triangulated categories of singularities. *math.AG/0503632*, 2005.
- [7] A. Takahashi. Matrix factorizations and representations of quivers I. *math.AG/0506347*, 2005.

Kazushi Ueda

Department of Mathematics, Graduate School of Science, Osaka University, Machikaneyama 1-1, Toyonaka, Osaka, 560-0043, Japan.

e-mail address : `kazushi@math.sci.osaka-u.ac.jp`